ON COVERINGS OF FOUR-SPACE BY SPHERES[1]

BY

F. L. CLEAVER

Let E_n denote Euclidean n-space and S_n the unit hypersphere about the origin. A *lattice* L in E_n is defined to be the totality of all points of the form $g_1X_1 + g_2X_2 + \cdots + g_nX_n$ where $g_1, g_2, \cdots, g_n$ are arbitrary integers and $X_1, X_2, \cdots, X_n$ is a fixed set of linearly independent points of E_n. The set $X_1, X_2, \cdots, X_n$ is called a *basis* of L. The *determinant* of L, written $d(L)$, is defined to be the determinant of the basis $X_1, X_2, \cdots, X_n$. L is said to be *S-admissible*, for a set S in E_n, if S contains no points of L in its interior other than the origin. Define the critical determinant $\Delta(S)$ to be $\inf\{|d(L)| : L$ is an S-admissible lattice$\}$, if there exists at least one S-admissible lattice, and ∞ otherwise. If $\Delta(S)$ is finite and if there exists an S-admissible lattice L such that $d(L) = \Delta(S)$, then L is called a *critical lattice* of S.

Call a system of nonoverlapping spheres a *regular packing* of spheres if their centers form a lattice; and call it a *semiregular packing* if their centers form the union of a lattice with a translation of the lattice, which does not form a new lattice. The *density* of a regular lattice packing is the volume of the sphere divided by the determinant of the lattice. The density of a semiregular packing is twice the volume of the sphere divided by the determinant of the lattice.

In this paper the following theorem is proved:

THEOREM 1. *Let $\mathscr{U}$ denote the set of lattices L in E_4 where L is S_4-admissible and for which there exists a unit hypersphere with no points of L in its interior, and let $s = \inf\{|d(L)| : L \in \mathscr{U}\}$, then $s = 1$. Moreover, if $L \in \mathscr{U}$ and if $d(L) = 1$, then either L is a unit cubic lattice where the unit hypersphere has its center at the center of one of the cells of L, or, for some choice of coordinates, L is generated by the points $(1,0,0,0)$, $(1/2, \sqrt{3}/2, 0, 0)$, $(0, 1/\sqrt{3}, \sqrt{2}/\sqrt{3}, 0)$, $(0,0,0,\sqrt{2})$ and the unit hypersphere has its center congruent to the point $(0, -1/\sqrt{3}, 1/\sqrt{6}, 1/\sqrt{2})$ with respect to L and has twelve points of L on its boundary.*

The second critical lattice is obtained by placing the critical lattice for S_3 in two hyperplanes a distance $\sqrt{2}$ apart.

Received by the editors April 3, 1961.

(1) This paper is part of the author's Tulane Dissertation in Geometry of Numbers, and was in part supported by the National Science Foundation. I wish to express my gratitude to Professor A. C. Woods for his help and encouragement.

Theorem 1 is easily seen to be equivalent to the following statement:

THEOREM A. *If L is an S_4-admissible lattice in E_4 and if $|d(L)| \leqq 1$, then any hypersphere H of radius greater than or equal to 1 must contain a point of L in its interior or on its boundary. If the radius of H is 1 and if H has no points of L in its interior, then $d(L) = 1$ and either L is the unit cubic lattice and H has its center C at the center of one of the cells of L, or, for some choice of coordinates, L is generated by the points $(1,0,0,0)$, $(1/2,\sqrt{3}/2,0,0)$, $(0,1/\sqrt{3},\sqrt{2}/\sqrt{3},0)$, $(0,0,0,\sqrt{2})$ and C is congruent to $(0,-1/\sqrt{3},1/\sqrt{6},1/\sqrt{2})$ with respect to L and H has twelve points of L on its boundary.*

THEOREM B. *Let L denote a lattice in E_4 with the following properties:*

(i) *L contains five points O, X_1, X_2, X_3, X_4 such that $OX_1 = OX_2 = OX_3 = OX_4$ and the five points do not lie in a three space.*

(ii) *$OX \geqq OX_1$ for every point X of L other than O. Then every hypersphere of radius $(|d(L)|)^{1/4}$ contains a point L in its interior or on its boundary. The point needs to be on the boundary only in the case where L is a rectangular cubic lattice and the hypersphere has its center at the center of one of the cells of L.*

Theorem B was first proved by Hofreiter [1], although his proof depends upon a large enumeration of cases and its correctness has probably not been thoroughly investigated. Dyson [2] gave a proof of this theorem in his famous paper in which he proved Minkowski's conjecture for four nonhomogeneous linear forms. This theorem is one of two which establish the conjecture.

Theorem 1 is a generalization of Theorem B. To see that this is so we show that Theorem 1 implies Theorem B. Suppose that L is a lattice satisfying the hypotheses of Theorem B. By expanding or contracting L without altering its shape, the truth or falsehood of Theorem B is not affected. Therefore we may assume that $d(L) = 1$. Let $q = OX_1$. The volume of the parallelepiped of which OX_1, OX_2, OX_3, OX_4 are edges is an integral multiple of the determinant of L. Therefore $q \geqq 1$. Thus, L is S_4-admissible and we have the hypotheses of Theorem A satisfied. Thus, any hypersphere H of radius 1 must contain a lattice point in its interior or on its boundary. In the critical case L is the unit cubic lattice since hypothesis (i) rules out the other critical lattice, and the implication holds.

Theorem 1 can be considered in another light. Let K denote the body consisting of S_4 and a unit hypersphere H with center at the end of one of the diameters of S_4 and the reflection of H. Then Theorem 1 shows that the critical determinant of K is 1 and that there are two critical lattices for K.

Theorem 1 also implies that the density of any semiregular lattice packing does not exceed the density of the densest regular packing. To show this we assume the proposition is not true. It is known that the densest regular lattice packing is by hyperspheres of radius $\frac{1}{2}$ with a lattice determinant of $\frac{1}{2}$. Now there exists a semiregular packing with lattice L and hyperspheres of radius $\frac{1}{2}$ having greater density.

This implies that $d(L) < 1$. Since about every lattice point we have nonoverlapping hyperspheres of radius $\frac{1}{2}$ the distance between lattice points must be at least 1, therefore L is S_4-admissible. Let S denote a hypersphere of radius $\frac{1}{2}$ with center a translated lattice point, then the distance from the center to any lattice point is at least 1. Now expand S to a hypersphere of radius 1 with the same center. Then S contains no lattice points in its interior. But this contradicts Theorem 1 and the proposition holds.

At the end of this paper an example is given that shows that the 5-dimensional analog of Theorem 1 is not true.

We now establish some preliminary results before beginning the proof of Theorem 1. Throughout we use Cartesian coordinates x, y, z, w.

The critical lattice of S_4 is known to be obtainable by rotating about O the lattice L generated by the points (1,0,0,0), (0,1,0,0), (0,0,1,0), and $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The 3-dimensional sublattice L_0, generated by the first three points, is the 3-dimensional unit cubic lattice and has six points on S_3, the 3-dimensional cross section of S_4. In the hyperplane $w = \frac{1}{2}$, L has eight points on the cross section of S_4, and in the hyperplane $w = 1$ the cross section is the single lattice point (0,0,0,1).

If we consider any 3-dimensional sublattice L_1 of L different from L_0, then the hyperplane containing L_1 has twelve points of L_1 on the cross section of S_4. We can choose a new set of rectangular coordinates so that L_1 is generated by $X_1 = (1,0,0,0)$, $X_2 = (1/2, \sqrt{3}/2, 0, 0)$, $X_3 = (0, 1/\sqrt{3}, \sqrt{2}/\sqrt{3}, 0)$ and L is generated by X_1, X_2, X_3, and $X_4 = (0, 1/\sqrt{3}, -1/\sqrt{6}, 1/\sqrt{2})$. Considering cross sections, we have a cross section of radius $1/\sqrt{2}$ in the hyperplane $w = 1/\sqrt{2}$ and six points of L in a rhombus shaped configuration. We have a similar cross section in the hyperplane $w = -1/\sqrt{2}$.

LEMMA 1. *Let L denote a lattice in E_3 such that $d(L) \leqq 1$ and L is S_3-admissible; then any sphere H of radius $(4[d(L)]^2 - 1)^{1/2}/2d(L)$ contains a point of L in its interior or on its boundary. Moreover, if H has no points of L in its interior and if $d(L) = 1$, then L is the unit cubic lattice, H has radius $\sqrt{3}/2$, and its center is the center of a cell of L. If H has no points of L in its interior and if $d(L) < 1$, then $d(L) = 1/\sqrt{2}$, L is generated by $X_1 = (1,0,0)$, $X_2 = (1/2, \sqrt{3}/2, 0)$, and $X_3 = (0, 1/\sqrt{3}, \sqrt{2}/\sqrt{3})$ for some choice of coordinates, and H has radius $1/\sqrt{2}$ and six points of L on its boundary.*

Proof. Let L be a lattice in E_3 with $d(L) \leqq 1$ and assume L to be S_3-admissible. Suppose there exists a sphere H of radius $r = (4[d(L)]^2 - 1)^{1/2}/2d(L)$ containing no points of L in its interior or on its boundary. Let (a,b,c) denote the center of H, and let $T(x,y,z) = (x-a, y-b, z-c, 1/2d(L))$. Then T maps E_3 onto the hyperplane $w = 1/2d(L)$ in E_4. Let $T(H) = H'$ be a sphere of equal radius and center $(0,0,0,1/2d(L))$. Let $T(L) = L_0$, then L_0 has no points inside or on H'.

Choose a basis X_1, X_2, X_3 of L and let X_4 denote a point of L_0 a minimum distance from $(0,0,0,1/2d(L))$. Let L' denote the 4-dimensional lattice generated by X_1, X_2, X_3 and X_4, where the fourth coordinates of X_1, X_2 and X_3 are now zero. Then $d(L') = d(L)1/2d(L) = \frac{1}{2}$. Since $1/2d(L) \geqq \frac{1}{2}$, the construction of L' assures us that L' is S_4-admissible. Moreover, H' is the cross section of S_4 in the hyperplane $w = 1/2d(L)$. But $\Delta(S_4) = \frac{1}{2}$. Therefore L' is the critical lattice for S_4. Since there can be at most twelve points of L' on S_4 in the hyperplane $w = 0$, there must be points of L' on H', for the lattice L' has 24 points on S_4. But this contradicts the fact that H' contains no points of L', therefore any sphere of the given radius must contain a point of L.

If we assume that H has no points of L in its interior and $d(L) = 1$, then H has radius $\sqrt{3}/2$ and we can construct L' as above and we obtain the critical lattice where L_1 is the unit cubic sublattice in the hyperplane $w = 0$, and H' has center $(0,0,0,\frac{1}{2})$. Thus L is the unit cubic lattice and H has for its center the center of a cell of L, in view of the discussion of the critical lattice of S_4 given above.

If we assume that H has no points of L in its interior and that $d(L) < 1$, again we construct L' as above. Since $1/2d(L) > \frac{1}{2}$, L' has exactly three hyperplanes meeting S_4. From the above discussion of the critical lattice of S_4, we know the sublattice L_1 in the hyperplane $w = 0$ must be generated by $X_1 = (1,0,0)$, $X_2 = (1/2, \sqrt{3}/2, 0)$, and $X_3 = (0, 1/\sqrt{3}, \sqrt{2}/\sqrt{3})$ for some choice of coordinates. Therefore L must also be this lattice. Moreover, $d(L) = 1/\sqrt{2}$ and the radius of H' is $1/\sqrt{2}$ in the hyperplane $w = 1/\sqrt{2}$. Therefore, H has six points of L on its boundary. This completes the proof of the lemma.

Proof of Theorem 1. Let $L \in \mathscr{U}$, then $d(L) \geqq \Delta(S_4) = \frac{1}{2}$. Therefore $s \geqq \frac{1}{2}$. Moreover, the unit cubic lattice with the unit hypersphere having its center at the center of one of the cells is in $\mathscr{U}$ and its determinant is 1. Therefore, $s \leqq 1$.

Next we show that there is a lattice L in $\mathscr{U}$ and a unit hypersphere H such that L is a critical lattice for the body consisting of S_4, H, and the reflection of H through O. Let $L \in \mathscr{U}$ and let S denote the unit hypersphere with center C. There are two cases to consider: either S has a point of L on its boundary or it does not. If S does not have a point of L on its boundary, then increase the radius until it does and select a point of L on S and call this point the origin O. Now decrease the radius of S again to 1 keeping the origin on the boundary of S and such that this unit hypersphere say S', is contained in the expanded S. Let K denote the body consisting of S_4, S' and the reflection of S' through O, then L is K admissible. If S contained a point of L on its boundary, then increasing and decreasing the radius is not necessary. Let K' denote the body consisting of S_4, a unit hypersphere H with center $(0,0,0,1)$ and the reflection of H through O. By rotating the lattice L to L' such that K goes onto K' we have a lattice L' that is K' admissible. Thus every lattice in $\mathscr{U}$ is K'-admissible for some rotation of the lattice. By Mahler's Compactness Theorem [3] the body K' has a critical lattice L. Moreover L is in $\mathscr{U}$ and $d(L) = s$. For the rest of the proof let L denote this fixed critical lattice and assume that $s < 1$. We then obtain a contradiction to this assumption which implies that $s = 1$.

Next we obtain a lower bound on the number of points of L on the boundary of K'. To do this we need a slight generalization of a theorem of Swinnerton-Dyer.

Let Q denote a bounded closed convex body in n-dimensions symmetric about the origin. Let Λ denote any Q-admissible lattice generated by the points $X_1, X_2, \cdots, X_n$; then define a lattice Λ' to lie in a small neighborhood of Λ if Λ' can be generated by a set of points $X'_1, X'_2, \cdots, X'_n$ each of which lies in a small neighborhood of the corresponding X_i. Call Λ extremal if in a sufficiently small neighborhood of Λ there are no Q-admissible lattices Λ' with $d(\Lambda') < d(\Lambda)$. Thus all critical lattices are extremal.

Swinnerton-Dyer [4] has proved that any extremal lattice Λ, with respect to the convex body Q, contains at least $n(n+1)/2$ point-pairs on the boundary of Q where a point-pair is a point and its reflection through O.

Let R denote the body in E_n consisting of a hypersphere T of radius r about the origin, a hypersphere S of radius r with center $(0,0,\cdots,0,r)$, and the reflection S' of S through O. Swinnerton-Dyer's theorem is also valid for this body provided that we count a point that occurs in the intersection of S and T or S' and T twice. The only change necessary in Swinnerton-Dyer's proof is that if P occurs in the intersection of S and T or S' and T we put two tac-planes through P, one with respect to T and one with respect to S or S'.

Applying this result to the body K' defined above, we have at least ten point-pairs of L on the boundary of K' with the possibility that some are counted twice.

If S_4 has three or more linearly independent points of L on its boundary, say X_1, X_2, X_3, then let L_0 denote the 3-dimensional sublattice generated by these points and let Q denote the hyperplane containing L_0. To show that the points X_1, X_2, X_3 generate a 3-dimensional sublattice L_0 of L we suppose that they do not. Then we have $1/\sqrt{2} \leqq d(L_0) = |\{X_1, X_2, X_3\}|/n$ where $1/\sqrt{2} = \Delta(S_3)$ and $|\{X_1, X_2, X_3\}|$ is the determinant of the three points and $n > 1$ is the index of the points with respect to L_0. But $|\{X_1, X_2, X_3\}| \leqq 1$, therefore $n \leqq \sqrt{2}$ which implies that $n = 1$, a contradiction, and the three points are a basis for L_0. Let Q' denote the hyperplane parallel to Q at a distance d from Q containing the closest lattice points of L to Q. There exists a point C' congruent to the center of H such that H', the unit hypersphere center C', meets Q in a sphere of radius r where $r \geqq (1 - d^2/4)^{1/2}$. Applying Lemma 1, we have $r \leqq (1 - 1/4[d(L_0))^2]^{1/2}$ since $d(L_0) \leqq 1$. This implies that $1 \leqq d \cdot d(L_0) = d(L) < 1$, a contradiction; therefore S_4 must have fewer than three linearly independent points of L on its boundary.

Note that the above argument holds for any 3-dimensional sublattice of L that has determinant less than or equal to 1. Therefore we may assume that every 3-dimensional sublattice of L must have determinant greater than 1.

Since S_4 has fewer than three linearly independent points of L on its boundary,

S_4 can have at most three point-pairs of L on its boundary, since the critical lattice for S_2 has three point-pairs on its boundary. Therefore, we must have at least seven distinct point-pairs on K' and counting the origin we have at least eight points of L on the hypersphere H.

Next we show that there must be at least nine points of L on H. If we have three point-pairs on S_4 and if we do not have three linearly independent points among them, then the six points must lie in a 2-dimensional cross section of S_4. Let Λ_0 denote the 2-dimensional sublattice generated by two of the points, say X_1 and X_2, where X_1 and X_2 are linearly independent. Then Λ_0 is the critical lattice for S_2 and its determinant is $\sqrt{3}/2$. Let Y be a point of L a minimal distance from the two-space containing Λ_0 and let c denote this distance. Let L_0 denote the 3-dimensional sublattice generated by X_1, X_2 and Y, then $d(L_0)=\sqrt{3}c/2$, and, since $d(L_0) > 1$, we have $c > 2/\sqrt{3}$. Let Z be a point of L a minimal distance from the three-space containing L_0 and let d denote this distance. Let q denote the distance from the projection of Z into the three-space containing L_0 to the two-space containing Λ_0. We want to show that $d^2 \geqq 3c^2/4$. Suppose that $d^2 < 3c^2/4$, then if $q \leqq c/2$ we have $(d^2 + q^2)^{1/2} < (3c^2/4 + c^2/4)^{1/2} = c$. But the distance from Z to the plane containing Λ_0 is $(d^2 + q^2)^{1/2}$ which is less than c contrary to our choice of Y. If $q > c/2$, then $(c-q)^2 < c^2/4$ and $(d^2 + (c-q)^2)^{1/2} < (3c^2/4 + c^2/4)^{1/2} = c$. But the distance from the lattice point $Z - Y$ to the plane containing Λ_0 is $(d^2 + (c-q)^2)^{1/2}$ which is less than c contrary to our choice of Y. Therefore $d \geqq \sqrt{3}c/2$. Now

$$d(L) = dc\,\frac{\sqrt{3}}{2} \geqq \frac{\sqrt{3}}{2}c^2\,\frac{\sqrt{3}}{2} = \frac{3}{4}\,c^2 > \frac{3}{4}\,\frac{4}{3} = 1,$$

but this is impossible since $d(L) < 1$; therefore we cannot have three point-pairs on S_4. Thus there are at most two point-pairs on S_4, and, counting the origin, we must have at least nine points of L on H.

Next we show that if there are eight points of L on H in a 3-dimensional cross section of H, then the theorem is true. We will use some results from a paper by the author [5]. According to Theorem 1 of this paper, we have a 3-dimensional rectangular sublattice of L. Let Q denote the hyperplane containing this cross section. Choose the origin O to be one of the eight points with $X_1 = (x,0,0,0)$, $X_2 = (0,y,0,0)$ and $X_3 = (0,0,z,0)$ so that we have a basis of the 3-dimensional sublattice. The remaining four points are given by $X_1 + X_2, X_1 + X_3, X_2 + X_3$, and $X_1 + X_2 + X_3$. By choosing the proper direction for the w-axis we have $C = (x/2, y/2, z/2, p)$ where C is the center of H and $p \geqq 0$. Let Q' denote the hyperplane parallel to Q in the positive direction of the w-axis that contains the closest lattice point of L. Let d denote the distance between Q and Q'. Let P_d denote the parallelepiped determined by the points $T = (0,0,0,d)$, $T + X_1$, $T + X_2, T + X_3, T + X_1 + X_2, T + X_1 + X_3, T + X_2 + X_3$ and $T + X_1 + X_2 + X_3$. Let F_1 denote the face of P_d determined by the points T, $T + X_1$, $T + X_2$ and

$T+X_1+X_2$; F_2 the face determined by T, $T+X_2$, $T+X_3$, $T+X_2+X_3$; and F_3 the face determined by T, $T+X_1$, $T+X_3$ and $T+X_1+X_3$. Let F_i' denote the face opposite F_i for $i=1,2,3$.

Let P_0 denote the parallelepiped determined by O, X_1, X_2, X_3, X_1+X_2, X_1+X_3, X_2+X_3 and $X_1+X_2+X_3$. Let H_0 denote the cross section of H in Q and H_d the cross section of H in Q'. Place hyperspheres of radius 1 about each of the vertices of P_0, then each of these eight hyperspheres meets Q' in a sphere of radius $(1-d^2)^{1/2}$ with centers at the respective vertices of P_d. The radius of H_d is $(1-(d-p)^2)^{1/2}$ and the radius of H_0 is $(1-p^2)^{1/2}$; also $|C|=1$, therefore $x^2/4+y^2/4+z^2/4+p^2=1$. Moreover, $d(L)=wyzd<1$ and $wyz>1$, therefore $d<1$.

Lemma 2. *The face F_1 must be covered by the sphere H_d and the four spheres of radius $(1-d^2)^{1/2}$ about each of the vertices of F_1, where covered means that every point of F_1 is interior to or on the boundary of at least one of the five spheres and there exists a point of F_1 that is not interior to any of the five spheres.*

Proof. Suppose first that F_1 is not covered, then decrease d until F_1 is covered. Now we show that H_d meets F_1 in a circle of radius r where $0\leqq r<(x^2/4+y^2/4)^{1/2}$. If $(1-(d-p)^2)^{1/2}<z/2$, then $(1-d^2)^{1/2}=(x^2/4+y^2/4)^{1/2}$ and $1-d^2+2pd-p^2<z^2/4$. Thus $x^2/4+y^2/4+2pd<z^2/4+p^2 = 1-x^2/4-y^2/4$ which implies that $x^2/2+y^2/2+2pd<1$, but this is impossible since $x\geqq 1$ and $y\geqq 1$ and $2pd\geqq 0$. If $r\geqq(x^2/4+y^2/4)^{1/2}$, then $(1-d^2)^{1/2}=0$ and $d=1$, a contradiction.

Since F_1 is covered, there exists a point X_4 in F_1 such that X_4 is on the sphere with center T and on H_d and X_4 is not interior to any of the five spheres. Therefore, $X_4=(tx,sy,0,d)$ where $0\leqq t\leqq \frac{1}{2}$ and $0\leqq s\leqq\frac{1}{2}$. From the symmetry of the covering of F_1 we have the following possible locations for X_4:

(1) $t=\frac{1}{2}$ and $0\leqq s\leqq\frac{1}{2}$,

(2) $t=0$ and $0<s\leqq\frac{1}{2}$,

(3) $0<t<\frac{1}{2}$ and $s=0$,

(4) $0<t<\frac{1}{2}$ and $s=\frac{1}{2}$.

We want to show that $d\geqq 1/\sqrt{2}$. Since X_4 is on the sphere about T we have $t^2x^2+s^2y^2=1-d^2$, and since X_4 is on H_d we have $(tx-x/2)^2+(sy-y/2)^2+z^2/4=1-(d-p)^2$. Squaring and substituting we have $tx^2+sy^2=1-2pd$.

If (1) holds, then $x^2/2+sy^2=1-2pd$ and $x^2/4+s^2y^2=1-d^2$. Eliminating x terms, we have $d^2-pd+(s^2y^2-sy^2/2-1/2)=0$, and, solving for d, we obtain $d=\frac{1}{2}[p+(p^2+2+2sy^2(1-2s))^{1/2}]$. Suppose that $d<1/\sqrt{2}$ then $(p^2+2+2sy^2(1-2s))^{1/2}<\sqrt{2}-p$ and squaring we have $sy^2(1-2s)<-\sqrt{2}p\leqq 0$. But $0\leqq sy^2(1-2s)$ since $0\leqq s\leqq\frac{1}{2}$ and we have a contradiction.

If (2) holds, then $s^2y^2=1-d^2$ and $sy^2=1-2pd$; moreover, we have $(1-d^2)^{1/2}\leqq y/2$. Suppose $d<1/\sqrt{2}$, then $1/\sqrt{2}<(1-d^2)^{1/2}\leqq y/2$ which implies that $y > \sqrt{2}$. Now $\frac{1}{2} > d^2 = 1 - s^2y^2$ thus $y > 1/\sqrt{2}s = y^2/\sqrt{2}(1-2pd)$ since $s = (1-2pd)/y^2$. Therefore $\sqrt{2} \geqq \sqrt{2}(1-2pd) > y$, since $0 < 1-2pd \leqq 1$, and we have a contradiction.

If (3) holds, then $tx^2 = 1 - 2pd$ and $t^2x^2 = 1 - d^2$, also $(1-d^2)^{1/2} \leqq x/2$. Suppose $d < 1/\sqrt{2}$, then $1/\sqrt{2} < (1-d^2)^{1/2} \leqq x/2$ and we have $\sqrt{2} < x$. Now $\frac{1}{2} > d^2 = 1 - t^2x^2$, and therefore $x > 1/\sqrt{2}t = x^2/\sqrt{2}(1-2pd)$. Thus we have $\sqrt{2} \geqq \sqrt{2}(1-2pd) > x$, since $0 < 1 - 2pd \leqq 1$, a contradiction.

If (4) holds then $tx^2 + y^2/2 = 1 - 2pd$ and $t^2x^2 + y^2/4 = 1 - d^2$. Eliminating the y terms and solving for d, we have $d = \frac{1}{2}[p + (p^2 + 2 + 2tx^2(1-2t))^{1/2}]$. Suppose $d < 1/\sqrt{2}$, then, squaring the inequality as in (1), we have $tx^2(1-2t) < -\sqrt{2}p$. But $0 < 1 - 2t < 1$ therefore $0 < -\sqrt{2}p \leqq 0$, a contradiction. Thus we have $d \geqq 1/\sqrt{2}$.

Let L' denote the lattice generated by X_1, X_2, X_3 and X_4, then $d(L') < d(L)$ since we have decreased d. If we can show that (a) L' is S_4-admissible and (b) that L' has no points interior to H, then we have a contradiction and F_1 must be covered.

Proof of (a). Since $d \geqq 1/\sqrt{2}$ it is sufficient to show that no points of $L' \cap Q'$ are interior to the sphere about T. Suppose that in the hyperplane Q' there exists a point Z of L' such that $|Z| < 1$. Then there exist integers m, n and u such that $Z = mX_1 + nX_2 + uX_3 + X_4$. Since $z \geqq 1$, it is sufficient to find a contradiction for $u = 0$; therefore

$$Z^2 = (mx + tx)^2 + (ny + sy)^2 + d^2 < 1$$

and

$$m^2x^2 + 2mtx^2 + n^2y^2 + 2nsy^2 + t^2x^2 + s^2y^2 < 1$$

and

$$(m^2 + 2mt)x^2 + (n^2 + 2ns)y^2 < 0.$$

However, for all integers m and n we have $(m^2 + 2mt) \geqq 0$ and $n^2 + 2ns \geqq 0$ since $0 \leqq s \leqq \frac{1}{2}$ and $0 \leqq t \leqq \frac{1}{2}$, and we have a contradiction.

Proof of (b). If $p = 0$, then $d \geqq 1/\sqrt{2}$ implies that $w = -d$, $w = 0$ and $w = d$ are the only hyperplanes meeting H. If $d = 2p$ where $p \neq 0$, then $(1-d^2)^{1/2} = 0$ and $d = 1$ which is impossible. If $d < 2p$, then the radius of H_d is greater than the radius of H_0 and F_1 is interior to H_d which again is impossible; therefore $d > 2p$. If $p \geqq \frac{1}{2}$, then $d > 2p \geqq 1$ which is impossible; therefore $0 \leqq p < \frac{1}{2}$. If $1 - 1/\sqrt{2} < p < \frac{1}{2}$, then $w = 0$ and $w = d$ are the only hyperplanes meeting H. If $0 < p \leqq 1 - 1/\sqrt{2}$, then $w = -d$, $w = 0$ and $w = d$ are the only hyperplanes meeting H. For if $w = 2d$ meets H, then $p \geqq 2/\sqrt{(2)} - 1$, but $2/\sqrt{(2)} - 1 > 1 - 1/\sqrt{2}$, a contradiction. Therefore to show that H has no points of L in its interior it is sufficient to check the hyperplanes $w = -d$ and $w = d$.

Suppose there exists a point Z of L' interior to H_d, then there exist integers m, n, and u such that $Z = mX_1 + nX_2 + uX_3 + X_4$ and we have

$$\left(mx + tx - \frac{x}{2}\right)^2 + \left(ny + sy - \frac{y}{2}\right)^2 + \left(uz - \frac{z}{2}\right)^2 < 1 - (d - p)^2 .$$

Since $z \geqq 1$, it is sufficient to check the planes $u = 0$ and $u = 1$ and since x and y are greater than or equal to 1 for each u, it is sufficient to check values of m and n between -1 and 1.

If $u = 0$ we have, after the usual substitutions

$$(m^2 + 2mt - m - t)x^2 + (n^2 + 2ns - n - s)y^2 < -1 + 2pd = -tx^2 - sy^2,$$

and

$$(m^2 + 2mt - m)x^2 + (n^2 + 2ns - n)y^2 < 0.$$

Again $m^2 + 2mt - m \geqq 0$ for all m and similarly for the coefficient of y^2 and we have a contradiction.

If $u = 1$ we obtain exactly the same inequality as above.

Suppose now in the hyperplane $w = -d$ there is a point Z of L' interior to H. Then $Z = mX_1 + nX_2 + uX_3 - X_4$, and

$$\left(mx - tx - \frac{x}{2}\right)^2 + \left(ny - sy - \frac{y}{2}\right)^2 + \left(uz - \frac{z}{2}\right)^2 < 1 - (d + p)^2 .$$

Again it is sufficient to check for $u = 0, 1$ and for each u, m and n between -1 and 1. After substituting, we have for $u = 0$, $(m^2 - 2mt - m + t)x^2 + (n^2 - 2ns - n + s)y^2 + 1 + 2pd < 0$. Now $1 = 2pd + tx^2 + sy^2$ therefore $(m^2 - 2mt - m + 2t)x^2 + (n^2 - 2ns - n + 2s)y^2 + 4pd < 0$. For $m = -1, 0, 1$ and $n = -1, 0, 1$ the coefficients of x^2 and y^2 are non-negative and we have a contradiction.

If $u = 1$, we obtain the same inequality as above. This completes the proof of (b).

Suppose now that the face F_1 is interior to the five spheres. We again want to obtain a contradiction so that we will have F_1 just covered and the lemma will be proved. We again want to show that H_d meets F_1 in a circle of radius r where $r \geqq 0$. Suppose that $r < 0$, then $(1 - (d - p)^2)^{1/2} < z/2$ and $1 - d^2 > x^2/4 + y^2/4$ since F_1 is interior to the four spheres. Thus $1 - d^2 + 2pd - p^2 < z^2/4$ and $x^2/4 + y^2/4 + 2pd < z^2/4 + p^2 = 1 - x^2/4 - y^2/4$. Therefore $x^2/2 + y^2/2 + 2pd < 1$, but this is impossible since $x \geqq 1$ and $y \geqq 1$ and $2pd \geqq 0$.

There exists a point Y of L in P_d. Since F_1 is interior to the five spheres, every face of P_d is interior to its respective five spheres and P_d is completely interior to the nine spheres. This is impossible, for then the lattice point Y of L would have to be interior to at least one of the spheres. This concludes the proof of the lemma.

In view of Lemma 2, the point of L in P_d must occur on the boundary of P_d. Suppose first that the point X_4 of L is in the face F_3 of P_d. Then using Theorem 1 from [5], X_1, X_3 and X_4 generate a 3-dimensional sublattice M of L and $d(M) > 1$. Since X_2 is perpendicular to the hyperplane containing M, we have $d(L) = d(M)y > 1$, a contradiction. If X_4 is in F_3', then $X_4 - X_2$ is in F_3, and a similar argument holds. Suppose next that X_4 is in F_1, then X_1, X_2 and X_4 generate a 3-dimensional sublattice M and $d(M) > 1$. But X_3 is perpendicular to the hyperplane containing M, therefore $d(L) = d(M)z > 1$ and we have a contradiction. If X_4 is in F_1', then $X_4 - X_3$ is in F_1 and the same argument holds. If X_4 is in F_2, then X_2, X_3 and X_4 again generate a sublattice M with $d(M) > 1$ and $d(L) = d(M)x > 1$ giving a contradiction. If X_4 is in F_2', then $X_4 - X_1$ is in F_2 and the same argument holds.

Thus we have shown that if we have eight points of $L \cap H$ in a hyperplane then the theorem is true and $d(L) = 1$.

LEMMA 3. *Let S be a hypersphere of radius* 1 *and Λ a lattice in E_4 such that Λ has no points interior to S and such that all the points of $\Lambda \cap S$ are in two parallel hyperplanes Q and Q'. Assume the origin is O and a 3-dimensional sublattice Λ_0 of Λ is in Q, and suppose the center C of S is between Q and Q' and that the distance between Q and Q' is d. Assume there are no lattice points between Q and Q'. Then if $d < 1$, Λ_0 can be expanded so that points of the lattice are farther apart and d can be decreased to obtain a new lattice Λ' such that $d(\Lambda') < d(\Lambda)$ and so that Λ' has no points interior to S.*

Proof. Let r_1 denote the radius of $S \cap Q$, d_1 the distance from C to Q, r_2 the radius of $S \cap Q'$ and d_2 the distance from C to Q'. Increase the length of the basis vectors of Λ_0 by a positive ε. Then we can decrease d_1 to d_1' and d_2 to d_2' with no lattice points interior to S. Then $d_1'^2 + r_1^2(1+\varepsilon)^2 = 1$, $d_1'^2 = 1 - (1 - d_1^2)(1+\varepsilon)^2$, and $d_1' = (d_1^2(1+\varepsilon)^2 - (2\varepsilon + \varepsilon^2))^{1/2}$. Using a Taylor's series expansion we have $d_1' = d_1(1+\varepsilon) - \varepsilon/d_1(1+\varepsilon) + o(\varepsilon^2)$; similarly we obtain $d_2' = d_2(1+\varepsilon) - \varepsilon/d_2(1+\varepsilon) + o(\varepsilon^2)$. Now $d' = d_1' + d_2' = d(1+\varepsilon) - \varepsilon d/d_1 d_2(1+\varepsilon) + o(\varepsilon^2)$. We have $d(\Lambda) = d(\Lambda_0)d$ and $d(\Lambda') = d(\Lambda_0)(1+\varepsilon)^3 d'$. Suppose for all $\varepsilon > 0$, $d(\Lambda_0)(1+\varepsilon)^3 d' \geqq d(\Lambda_0)d$. Then

$$(1+\varepsilon)^3\left[d(1+\varepsilon) - \frac{\varepsilon d}{d_1 d_2(1-\varepsilon)}\right] \geqq d.$$

This implies that $d_1 d_2(1+\varepsilon)^4 - \varepsilon(1+\varepsilon)^2 - d_1 d_2 \geqq 0$ and we have

$$4d_1 d_2 - 1 + d_1 d_2(6\varepsilon + 4\varepsilon^2 + \varepsilon^3) - (2\varepsilon + \varepsilon^2) \geqq 0$$

for all $\varepsilon > 0$. This implies that $4d_1 d_2 - 1 \geqq 0$ and $d_1 d_2 \geqq 1/4$. But $d_1 + d_2 = d < 1$, therefore $d_1 < 1 - d_2$ and $d_1 d_2 < d_2 - d_2^2$ for $0 < d_2 < 1$. But the function $d_2 - d_2^2$ has as absolute maximum of 1/4 for $d_2 = 1/2$, and we have a contradiction. Thus the lemma is proved.

We have proved that there are at least nine points of L on H; moreover, if there are exactly nine points, then we must have two linearly independent points a distance one from the origin. Now we apply Theorem 2 from [5], and if H has ten or more points of L, then there exists a hyperplane containing a rectangular sublattice and the theorem is proved. Therefore we assume that there are exactly nine points of L on H, then we have the nine points in parallel hyperplanes K and K' and in K we have the 6-configuration and in K' a congruent triangle or the reflected triangle. We observe that the center C of H must line a hyperplane between K and K', for if both K and K' meet H in a hemisphere, then we can move H in a direction perpendicular to K and away from, K, and expand H to obtain an inscribed hypersphere of larger radius than H, a contradiction.

First we consider the case where we have the congruent triangle in K'. Let d denote the distance between K and K'; then we show that $d > \frac{1}{2}$. Label the points in K as $O, X_1\ X_2, X_3, X_1+X_3, X_2+X_3$; then X_3 is perpendicular to X_1 and to X_2. The points in K' are given by X_4, $X_4 + X_1$ and $X_4 + X_2$ where X_4 is perpendicular to X_1. Let L_0 denote the 3-dimensional sublattice generated by O, X_1, X_3, and X_4. To make $d(L_0)$ as large as possible we assume that O, X_1, and X_2 form an equilateral triangle. Let x denote the length of the sides of this triangle; then $|X_1| = x$ and $|X_3| \leqq (2/\sqrt{3})(3-x^2)^{1/2}$. The distance from X_4 to the plane determined by O, X_1, X_3, is d, therefore $d(L_0) \leqq |X_1|\,|X_3|\,d \leqq (2xd/\sqrt{3})\,(3-x_2)^{1/2}$. Suppose that $d \leqq \frac{1}{2}$, then since $d(L_0) > 1$, we have $1 < (x/\sqrt{3})(3-x^2)^{1/2}$ which implies that $x^4 - 3x^2 + 3 < 0$. But $x^4 - 3x^2 + 3$ is a positive definite form, and we have a contradiction, thus $d > \frac{1}{2}$.

Chose the w-axis perpendicular to K with the positive direction toward K', then $w = 0$ $w = d$ are the hyperplanes K and K'. Since $d > \frac{1}{2}$, no point of L in $w = nd$ is within unit distance of O for $n \geqq 2$.

Let X_4' denote the projection of X_4 into K, then either the distance from X_4' to the plane containing O, X_1, X_2, is less than or equal to $\frac{1}{2}\,|X_3|$ or greater than this value. If it is greater, then we can rechoose O to be X_3 and X_3 to be O and relabel the points. Therefore without loss of generality we may assume the distance to be less than or equal to $\frac{1}{2}\,|X_3|$.

Suppose now that $|X_4| > 1$, then all the points of L in K' are a distance greater than 1 from any lattice point in K. Now $d < 1$, therefore we may apply Lemma 3 and obtain a lattice L' that has smaller determinant than L, is S_4-admissible and has no points interior to H. This is a contradiction, since L is critical, and we have $|X_4| = 1$.

Now $|X_4 - X_3| \geqq 1$, and suppose first that $|X_4 - X_3| > 1$. Let Q denote the the hyperplane containing O, X_1, X_2, X_4 then $X_1 + X_4$ and $X_4 + X_2$ are also in Q, and we have the 6-configuration since X_4 is perpendicular to X_1 and to X_2. Let Q' be the parallel hyperplane containing X_3, $X_3 + X_1$ and $X_3 + X_2$. If $|X_3| > 1$, since $|X_4 - X_3| > 1$, the lattice points of Q' are a distance greater than 1 from the lattice points in Q. Again the distance between Q and Q' is between $\frac{1}{2}$ and 1; therefore we can apply Lemma 3 and obtain a lattice L' with smaller determinant, that is S_4 admissible and has no points in H, a contradiction. Therefore $|X_3| = 1$. Let R denote the hyperplane containg O, X_2, X_3, X_4 then $X_2 + X_4$ and $X_2 + X_3$ are also in R, and we have the 6-configuration since X_2 perpendicular to X_4 and X_3. Let R' be the parallel hyperplane containing $X_1, X_1 + X_4$ and $X_1 + X_3$. Since H has exactly nine points, we must have exactly two linearly independent vectors of unit length, and they are X_4 and X_3 in R. Thus all the lattice points in R' are a distance greater than 1 from the lattice points in R. But again the distance between R and R' must be between $\frac{1}{2}$ and 1, therefore we can apply Lemma 3, as above, to obtain a contradiction. Therefore $|X_4 - X_3| = 1$. But again we have two linearly independent vectors X_4 and $X_4 - X_3$ in R, therefore the lattice points of R' are a distance greater than 1 from the lattice points of R. Applying Lemma 3 again, we have final contradiction and this case cannot arise.

Suppose now that the triangle in K' is the reflected triangle. Label the points in K as before; then the points in K' are given by $X_4, X_4 + X_2, X_4 + X_2 - X_1$. We wish to show that $d > 1/3$. Let L_0 denote the 3-dimensional sublattice generated by O, X_1, X_3 and X_4. To make $d(L_0)$ as large as possible we assume that O, X_1, and X_2 form an equilateral triangle, and let x denote the length of a side. Then $|X_1| = x$ and $|X_3| \leqq (2/\sqrt{3})(3 - x^2)^{1/2}$. The distance from X_4 to the plane containing O, X_1, X_3, and $X_1 + X_3$ is given by $(x^2/12 + d^2)^{1/2}$; therefore $d(L_0) \leqq |X_1||X_3|(x^2/12 + d^2)^{1/2}$ and $d(L_0) > 1$. Suppose that $d \leqq 1/3$, then $1 < (4/3)x^2(3 - x^2)(x^2/12 + 1/9)$, which implies that $3x^6 - 5x^4 - 12x^2 + 27 < 0$. Now $1 \leqq x$ and $1 \leqq |X_3| \leqq (2/\sqrt{3})(3 - x^2)^{1/2}$ and this implies that $x \leqq 3/2$. However, the function $f(x) = 3x^6 - 5x^4 - 12x^2 + 27$ is positive throughout the interval $1 \leqq x \leqq 3/2$, a contradiction; therefore $d > 1/3$. This means that there are no lattice points in the hyperplanes $w = nd$ for $n \geqq 3$ within unit distance of any lattice point in K.

We now show that there are no lattice points in the hyperplane $w = 2d$ within unit distance of a lattice point in K. We may assume that the distance from the projection of the triangle in K' into K to the plane containing O is q and that $q \leqq \frac{1}{2}|X_3|$. For if not, choose O to be X_3 and relabel the points in K. In K' we choose the point X_4 so that $|X_4 - X_1| \leqq |X_4 - X_2|$ and $|X_4 - X_1| \leqq |X_4|$. Now we list some equalities and inequalities needed in the argument.

(1) $X_k^2 = 2X\ C$ for $k = 1, 2, 3, 4$,

(2) $X_1X_3 = 0$,

(3) $X_2X_3 = 0$,

(4) $X_1^2 = X_1X_4$,

(5) $X_2^2 = X_2X_4$,

(6) $X^2 \leqq X_1^2$ since $(X_4 - X_1)^2 \leqq (X_4 - X_2)^2$,

(7) $X_2^2 \leqq 2X_1X_2$ since $(X_4 - X_1)^2 \leqq (X_4 - X_2 - X_1)^2$.

The distance q is given by $X_3X_4/|X_3| \leqq \frac{1}{2}|X_3|$, therefore

(8) $2X_3X_4 \leqq X_3^2$.

There are no lattice points on H in the hyperplane $w = -d$ so we have $(-X_4 + X_3 + X_2 - C)^2 > C^2$ which implies

(9) $X_4^2 > X_2^2 + X_3X_4$.

We also have $(-X_4 + X_3 + X_1 - C)^2 > C^2$; therefore

(10) $X_4^2 > X_1^2 + X_3X_4$.

$(-X_4 + X_3 + X_1 + X_2 - C)^2 > C^2$; therefore

(11) $X_4^2 + X_1X_2 > X_1^2 + X_2^2 + X_3X_4$.

Now we want to show that the lattice points in $w = 2d$ are not within unit distance of the origin. This is equivalent to showing that the distance from lattice points of the following form to O is greater than the distance from $X_4 - X_1$ to O:

(a) $2X_4 - X_1 - nX_2$ for $n \geqq 1$,

(b) $2X_4 - X_1 - (X_1 + nX_2)$ for $n \geqq 0$,

(c) $2X_4 - X_1 - nX_2 - X_3$ for $n \geqq 0$,

(d) $2X_4 - X_1 - (X_1 + nX_2) - X_3$ for $n \geqq 0$.

Suppose that $(2X_4 - X_1 - nX_2)^2 \leqq (X_4 - X_1)^2$, then squaring and simplifying we have $3X_4^2 - 2X_1^2 + (n^2 - 4n)X_2^2 + 2nX_1X_2 \leqq 0$. If $n = 1$, we have $3X_4^2 - 2X_1^2 - 3X_2^2 + 2X_1X_2 \leqq 0$. Then using (11) we have $X_4^2 - X_2^2 + X_3X_4 < 0$ and using (10) we obtain $X_1^2 - X_2^2 + 2X_3X_4 < 0$. But this contradicts (6), since Voronoï has shown that the lattice vectors from points on an inscribed hypersphere must make acute angles; therefore $X_3X_4 \geqq 0$. If $n \geqq 2$, we apply (11) to obtain $X_1^2 + (n^2 - 4n + 3)X_2^2 + 3X_3X_4 + (2n - 3)X_1X_2 < 0$, which gives us a contradiction since $X_1^2 \geqq X_2^2$.

For (b) we have $3X_4^2 - 3X_1^2 + (n^2 - 4n)X_2^2 + 4nX_1X_2 \leqq 0$. For $n = 0$ we have $X_4^2 \leqq X_1^2$ which contradicts (10). Applying (11) we have $(n^2 - 4n + 3)X_2^2 + 3X_3X_4 + (4n - 3)X_1X_2 < 0$. Clearly we have a contradiction if $n = 1$, or if $n \geqq 3$. If $n = 2$, we have $-X_2^2 + 2X_1X_2 + 3X_3X_4 + 3X_1X_2 < 0$ which contradicts (7).

For (c) we have, using (8), $3X_4^2 - 2X_1^2 + (n^2 - 4n)X_2^2 - 2X_3X_4 + 2nX_1X_2 \leqq 0$, and if $n = 0$, we have $3X_4^2 - 2X_1^2 - 2X_3X_4 \leqq 0$, and applying (10), we have $X_1^2 + X_3X_4 < 0$, a contradiction. If $n \geqq 1$, apply (11) to get $X_1^2 + (n^2 - 4n + 3)X_2^2 + (2n - 3)X_1X_2 + X_3X_4 < 0$. If $n \geqq 3$, we have a contradiction, and if $n = 1$, we have $X_1^2 - X_1X_2 + X_3X_4 < 0$ which implies that $X_1(X_1 - X_2) < 0$; but this implies an obtuse angle in the lattice points on H, a contradiction. If $n = 2$, we have $X_1^2 - X_2^2 + X_1X_2 + X_3X_4 < 0$ which again implies, using (7), that $X_1^2 - X_1X_2 + X_3X_4 < 0$, a contradiction.

For (d) we have, using (8), $3X_4^2 - 3X_1^2 + (n^2 - 4n)X_2^2 - 2X_3X_4 + 4nX_1X_2 \leqq 0$. If $n = 0$, we have $3X_4^2 - 3X_1^2 - 2X_3X_4 \leqq 0$, and using (10), we have $X_3X_4 < 0$, a contradiction. If $n \geqq 1$, using (11), we have $(n^2 - 4n + 3)X_2^2 + X_3X_4 + (4n - 3)X_1X_2 < 0$, and if $n \geqq 3$, we have a contradiction. If $n = 1$, we have $X_1X_2 - X_2^2 + X_3X_4 < 0$ which implies $X_2(X_1 - X_2) < 0$ and again we have an obtuse angle. If $n = 2$, we have $-X_2^2 + 5X_1X_2 + X_3X_4 < 0$ which implies an obtuse angle. This completes the list of possibilities.

Again we assume that the distance from the projection of the triangle in K' into K to the parallel plane containing the origin is less than or equal to the distance to the parallel plane containing X_3. Let π denote the plane containing O, X_1, and X_2, then we project the triangle into K and then into π. The vertices of the projected triangle lie in $\pi \cap H$, and these three points along with O, X_1, X_2 are the vertices of a hexagon inscribed in $\pi \cap H$. The adjacent vertices determine possibly three different distances. Let q denote the minimum of these three distances. The adjacent vertices, distance q apart, we now label one O and the other one the projection of X_4. The point opposite the projection of X_4 we label X_2. This determines the labelling of the remaining points, those in K' are $X_4, X_4 + X_2$, and $X_4 + X_2 - X_1$.

Suppose now that $|X_4| > 1$, then, since q is minimal, the lattice points in K' are a distance greater than 1 from those in K. Furthermore, we have shown that all the lattice points in the other hyperplanes are more than unit distance from O; therefore we may apply Lemma 3. Thus we obtain a lattice L' with smaller determinant that is S_4-admissible and with no points interior to H, which contradicts the fact that L is a critical lattice. Therefore $|X_4| = 1$.

Now $|X_4 - X_3| \geqq 1$ and suppose that $|X_4 - X_3| = 1$; then we have two linearly independent vectors of unit length in L. Since we have nine points on H, we must have exactly two such vectors. Let Q denote the hyperplane containing O, X_1, X_3, X_4, then $X_1 + X_3$ is also in Q, and $X_2, X_2 + X_3, X_2 + X_4$, and $X_2 + X_4 - X_1$ are in a parallel hyperplane Q'. Moreover, the two linearly independent vectors of length one are in Q; therefore all the lattice points of Q' are a distance greater than one from the lattice points in Q. Since the distance between Q and Q' is less than one, we apply Lemma 3 to obtain a lattice L', with smaller determinant, that is S_4-admissible, a contradiction. Therefore $|X_4 - X_3| > 1$.

Now suppose that q is strictly less than each of the other two distances. Then the distance between the lattice points, other than X_4, to the lattice points in K is greater than one. Suppose also that the projection of X_1 onto the line segment OX_2 is between the midpoint and O. If $|X_1| > 1$, let λ denote the line through X_1 parallel to OX_2 and move X_1 a small epsilon distance on the line λ toward O. Then $X_4 + X_2 - X_1$ moves the same distance in the opposite direction. For sufficiently small epsilon we have no new points of L on H and the distance between lattice points remains greater than or equal to one in all the hyperplanes. Thus we obtain a new lattice L' that is S_4-admissible with the same determinant as L and with no points interior to H. Therefore L' is a critical lattice, but L' has only six points on H since $X_1, X_1 + X_3$ and $X_4 + X_2 - X_1$ are no longer on H. This is contrary to the fact that a critical lattice must have at least nine points on H; therefore $|X_1| = 1$. Let Q be the hyperplane through O, X_1, X_2, and X_4 then $X_2 + X_4$ and $X_4 + X_2 - X_1$ are also in Q. The points $X_3, X_3 + X_2$, $X_3 + X_1$ lie in a parallel hyperplane Q'. Moreover, the two linearly independent vectors of unit length X_1 and X_4 are in Q. Then, as above, we may apply Lemma 3 and obtain an S_4-admissible lattice with no points interior to H and smaller determinant than L. This is a contradiction, therefore the projection of X_1 onto OX_2 must lie between the midpoint and X_2.

If $|X_2 - X_1| = 1$, let Q be the hyperplane through O, X_1, X_2, and X_4; then $X_2 + X_4$ and $X_4 + X_2 - X_1$ are also in Q. The points X_3, $X_3 + X_2$ and $X_3 + X_1$ lie in a parallel hyperplane Q'. Moreover, the two linearly independent vectors of unit length X_4 and $X_2 - X_1$ are in Q. Thus, as above, we apply Lemma 3 to obtain a contradiction; therefore $|X_2 - X_1| > 1$.

Again let λ denote the line through X_1 parallel to OX_2. Since the projection of X_1 onto OX_2 is between the midpoint and X_2 and since $|X_2 - X_1| > 1$, we move X_1 a small epsilon distance on λ away from O; then $X_4 + X_2 - X_1$ moves the same distance in the opposite direction. As above, for sufficiently small epsilon we have no new points on H, and the distance between lattice points remains greater than or equal to one in all the hyperplanes. Again we have a new critical lattice with only six points on H, a contradiction; therefore q must be equal to one or both of the distances in the hexagon.

Suppose first that q is equal to the distance from O to the projection of $X_4 + X_2 - X_1$ into π. Then we have $|X_4| = 1$ and $|X_4 + X_2 - X_1| = 1$. Let Q be the hyperplane containing O, X_1, X_2, X_4, $X_2 + X_4$ and $X_4 + X_2 - X_1$, then the points X_3, $X_3 + X_2$ and $X_3 + X_1$ are in a parallel hyperplane Q'. Moreover, the two linearly independent vectors of unit length are in Q; therefore the distance between lattice points of Q and those of Q' is greater than one. As above, we may apply Lemma 3 to obtain a contradiction.

Therefore q must be equal to the remaining distance in the hexagon which is the distance from the projection of $X_4 + X_2 - X_1$ to X_2. Thus $|X_4 + X_2 - X_1 - X_2| = |X_4 - X_1| = 1$ and we have $|X_4| = 1$. Again let Q be the hyperplane containing O, X_1, X_2, X_4, $X_2 + X_4$ and $X_4 + X_2 - X_1$. Thus the two linearly independent vectors of length one are in Q and the remaining three points in a parallel hyperplane Q'. As above we apply Lemma 3 to get the final contradiction; therefore this case cannot arise.

Thus we have exhausted all the possibilities for L and finally we have the fact that $d(L) = 1$.

We proceed now to a discussion of the critical lattices. Let $L \in \mathcal{U}$ such that that $d(L) = 1$. Then if every 3-dimensional sublattice of L has determinant greater than 1, then such a lattice was shown to be impossible in the proof thus far. Therefore there must exist a 3-dimensional sublattice L_0 of L such that $d(L_0) \leqq 1$. Let K denote the hyperplane containing L_0 and let d denote the distance to the closest parallel hyperplane K' containing a point of L.

If $d(L_0) = 1$, then $d(L_0)d = d(L) = 1$ and $d = 1$. There exists a point C' congruent to C with respect to L, where C is the center of H, such that the radius r of H_0 $(= H' \cap K)$ is greater than or equal to $(1 - d^2/4)^{1/2} = \sqrt{3}/2$. Applying

Lemma 1, we have that L_0 must be a unit cubic lattice and $r = \sqrt{3}/2$, therefore C' is in the hyperplane exactly half way between K and K'. Thus $H' \cap K' = H_1$, a sphere also of radius $\sqrt{3}/2$, and the lattice configuration in K' must be identical to that in K. Otherwise H_1 would have a point of L in its interior. Therefore L is the unit cubic lattice and C is the center of one of the cells of L.

If $d(L_0) < 1$, then $d = 1/d(L_0)$. As above there exists a unit hypersphere H' with center C' where C' is congruent to C modulo L such that the radius r of $H_0 = (H \cap K)$ is greater than or equal to $(1 - d^2/4)^{1/2} = (4d(L_0)^2 - 1)^{1/2}/2d(L_0)$. Moreover, H_0 has no points of L_0 in its interior; therefore we may apply Lemma 1. Thus $d(L_0) = 1/\sqrt{2}$ and for some choice of coordinates L_0 is generated by $X_1 = (1,0,0,0)$, $X_2 = (1/2, \sqrt{3}/2, 0, 0)$ and $X_3 = \sqrt{(0, 1/\sqrt{3}, \sqrt{2}/\sqrt{3}, 0)}$ and H_0 has radius $1/\sqrt{2}$ and has six points of L_0 on its boundary. Furthermore, $d = \sqrt{2}$ and $H_1 = H' \cap K'$ must also have radius $1/\sqrt{2}$. The lattice configuration in K' must be identical to that in K, otherwise H_1 would have a point of L in its interior. Therefore L is generated by X_1, X_2, X_3, and $X_4 = (0,0,0,\sqrt{2})$, and has six points on each of the cross sections in $w = 0$ and $w = \sqrt{2}$ hyperplanes. Moreover, C' is clearly congruent to $(0, -1/\sqrt{3}, \sqrt{(2)}/2\sqrt{3}, \sqrt{2}/2)$ modulo L. This completes the proof of Theorem 1.

The 5-dimensional analog of Theorem 1 with hypersphere H of radius $\sqrt{5}/2$ is not true. To show this, consider the critical lattice L_0 for S_4 generated by the points $(1,0,0,0)$, $(1/2, \sqrt{3}/2, 0, 0)$, $(0, 1/\sqrt{3}, \sqrt{2}/\sqrt{3}, 0)$, $(0, 1/\sqrt{3}, -1/\sqrt{6}, 1/\sqrt{2})$. Let S_0 denote the hypersphere of radius $\frac{1}{2}$ with center $(0, -1/\sqrt{3}, 1/\sqrt{6}, 0)$. Then S_0 has eight points of L_0 on its boundary and no points of L_0 on its interior. Let H be a hypersphere in E_5 with radius $\sqrt{5}/2$ and center $(0, -1/\sqrt{3}, 1/\sqrt{6}, 0, \sqrt{3}/2)$. Let L_0 be a 4-dimensional lattice in the hyperplane $r = 0$, where r denotes the fifth coordinate. Let L denote the 5-dimensional lattice generated by the above four basis points of L_0 and the point $(0,0,0,0,\sqrt{3})$. Then H has sixteen points of L on its boundary and no point of L in its interior and $d(L) = \frac{1}{2}\sqrt{3} < 1$. By expanding L without altering its shape and so that the determinant becomes 1, we obtain a lattice L' that is S_5 admissible with determinant 1 that has no points either interior or on the boundary of H.

Bibliography

1. N. Hofreiter, Monatsh. Math. Phys. **40** (1933), 351–406.

2. F. J. Dyson, *On the product of four non-homogeneous linear forms*, Ann. of Math. (2) **49** (1948), 2–109.

3. K. Mahler, *On lattice points in n-dimensional star bodies*, Proc. Roy. Soc. Ser. A **187** (1946), 157–187.

4. H. P. F. Swinnerton-Dyer, *Extermal lattices of convex bodies*, Proc. Cambridge Philos. Soc. **49** (1953), 161–162.

5. F. L. Cleaver, *On a theorem of Voronoï*, Trans. Amer. Math. Soc. **120** (1965), 390–400.

University of South Florida,
Tampa, Florida